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LETTER TO THE EDITOR

Percolative phase transition without the appearance of an infinite network

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Abstract. We study the probability P_n for the origin to belong to a cluster of n sites in a correlated site–bond percolation problem, the Coniglio–Stanley–Klein model. In some region of the phase diagram we obtain rigorously a percolative phase transition in the asymptotic behaviour of P_n, but without divergence of the mean cluster size and without the appearance of an infinite network.

In this Letter we study a correlated site–bond percolation problem. Consider plus and minus spins distributed according to an Ising model in thermal equilibrium; an additional randomness now is introduced by assuming that connecting bonds will be *active* with probability p_B, independently of each other, and *passive* with probability $1-p_B$. Clusters are then the maximal sets of plus spins connected through active bonds. (For the Hamiltonian and the magnetic properties, both the passive and the active bonds are taken into account; only for the definition of clusters do we distinguish between the two types of bonds.) This model was introduced by Coniglio *et al* (1979) in connection with polymer gelation; we regard the probability p_B to establish a chemical bond (i.e. to activate it) as an independent parameter, in addition to temperature and magnetic field. For simplicity we always assume a nearest-neighbour Ising model in zero magnetic field, and for temperatures below the critical temperature T_c we will consider only the two pure positive and negative phases. Some numerical results were given by Stauffer (1981)¶.

In this correlated site–bond percolation model the probability for the origin to belong to a cluster of exactly n plus spins is called P_n, and the probability for the origin to belong to an infinite cluster is called P_∞. Both depend on p_B, the temperature T and, for $T<T_c$, also on the phase of the system.

A particular case of this model, where

$$\ln(1-p_B) = (\text{exchange energy})/(\text{thermal energy})$$

was studied recently for another purpose by Coniglio and Klein (1980) and is currently simulated by Monte Carlo methods (Roussenq 1981, Ottavi 1981, Stauffer 1981).

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¶ Note that in figure 3 there T/T_c should be replaced be T_c/T.

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According to Coniglio and Klein it is those clusters with this particular choice of p_B, and not the usually studied clusters of the Ising model ($p_B = 1$), which are determined by the phase transition behaviour and which are relevant e.g. for nucleation theory.

For the asymptotic behaviour in usual percolation models of the cluster size distribution $P_n(p_B)$, Kunz and Souillard (1978) proved that this quantity undergoes a percolative phase transition from an exponential decay at low concentrations ($\ln P_n \propto n$) to a decay with a surface exponent at high concentrations ($\ln P_n \propto n^{1-1/d}$ in d dimensions). We refer to them (Aizenman *et al* 1980 or Stauffer 1979) for a discussion of motivations and more results. Aizenman *et al* (1980) showed for usual percolation and for Ising models with rather general interactions in $d \geqslant 2$ dimensions that

$$P_n \geqslant \exp(-\text{constant} \times n^{1-1/d}) \tag{1}$$

as soon as $P_\infty > 0$. Numerical evidence (Stauffer 1979) suggests that this inequality actually is an equality, as far as the leading behaviour of the argument of the exponential is concerned.

Now we want to know if the surface exponent $1-1/d$ of equation (1) is also valid when percolation occurs in the more general correlated site–bond percolation model discussed above. Our answer to this question is yes, but as we shall see below, as a result the asymptotic decay of the cluster size distribution presents in some regions of the phase diagram, a strange and new characteristic. First we present the following rigorous results, where α, α' and α'' are factors independent of n:

Theorem 1.

(i) If $p_B < \varepsilon$, with some fixed positive ε independent of T and of the phase, then $P_n < \exp(-\alpha)$.

(ii) If $P_\infty > 0$ then $P_n > \exp(-\alpha' n^{1-1/d})$.

(iii) If $P_\infty > 0$, for $T < T_c$ and some p_B in the positive pure phase, then also $P_n > \exp(-\alpha'' n^{1-1/d})$ for the negative pure phase.

(Parts (i) and (ii) are also valid in an arbitrary magnetic field.)

The proof of this theorem is an easy extension of the one by Aizenman *et al* (1980) and hence will not be given here. If we combine these results with a study of the phase diagram (Stauffer 1981) of this model and with some other results, we will get new behaviour as promised. Figure 1 shows this phase diagram in three dimensions for temperatures below T_c to which we are now restricting ourselves. We then can describe our system by the two variables p_B and $M = 2x - 1$, where M is the magnetisation and x the concentration of up spins. The boundaries $M = \pm 1$ correspond to the plus and minus phases at zero temperature, with all spins down for $M = -1$ (no clusters of up spins) and all spins up for $M = +1$ (pure bond percolation at p_B). The centre line $M = 0$ corresponds to $T = T_c$. Along the upper line, $p_B = 1$, we get the usual clusters of up spins in the plus or minus phase of the Ising model.

In the shadowed region an infinite cluster exists (see Stauffer 1981 for three and Ottavi 1981 for two dimensions). In fact one can prove rigorously that for M near $+1$ and for M near -1 the phase diagram is of the type described in our figure. The broken curve is the mirror image in the left part ($M < 0$) of the solid border line of the region where an infinite network occurs (shaded region in right part, $M > 0$). Thus any phase above the broken curve at $M < 0$ can coexist with a phase at the right-hand side (positive M) where $P_\infty > 0$. Hence according to theorem 1 we found a whole region (M

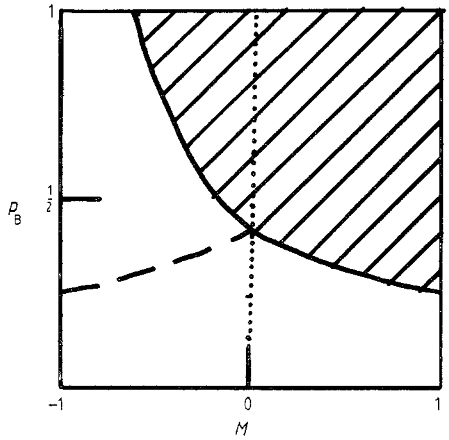

Figure 1. Schematic phase diagram for three dimensions. The broken curve is the mirror image of the border line of the shadowed region. An infinite network is present in the shadowed region. The broken curve presumably is the locus of a phase transition in the asymptotic cluster numbers without the appearance of an infinite network and without divergence of any moment of the cluster size distribution. The dotted line corresponds to $T = T_c$.

negative, all p_B above the broken curve until the shadowed region or unity) in which $P_n > \exp(-\alpha n^{1-1/d})$ although $P_\infty = 0$, 'surface' decay of P_n without the presence of an infinite network. On the other hand P_n decays as $\exp(-\alpha n)$ for small p_B (theorem 1), and one expects from renormalisation group arguments, similar to those of Klein and Stauffer (1980) that this simple exponential decay extends from $p_B = 0$ up to the broken curve. At this broken curve, therefore, the asymptotic decay of the cluster numbers changes from $\ln P_n \propto -n$ below to $\ln P_n \propto -n^{1-1/d}$ above the broken curve, for $n \to \infty$, without the appearance of an infinite cluster, if these expectations are correct. We cannot yet exclude rigorously that this border, for the simple exponential decay $\ln P_n \propto -n$, occurs at a p_B somewhere below the broken curve.

Wherever this transition in the asymptotic decay of cluster numbers is, we now ask if it is also characterised by a divergence of the first or any higher moment of the cluster size distribution P_n, as happens in usual percolation (Stauffer 1979). Our answer is no, at least for magnetisations close to −1. More precisely, we can prove an upper bound of the cluster size distribution in the minus phase at very low temperatures, using an analogous result derived by Delyon (1979) for the Ising model:

Theorem 2.
There is some ε' such that if $-1 \leqslant M \leqslant -1+\varepsilon'$ then $P_n < \exp(-\alpha'' n^{1-1/d})$ for all p_B between zero and unity.

This inequality prevents any moments from diverging.

Hence we have proved for very small concentrations of up spins that the transition in the asymptotic behaviour of P_n is not accompanied by an order parameter (infinite network), a divergence of the susceptibility (first moment of P_n) or any divergence in any of the higher moments of P_n. This description presumably applies to the whole region in figure 1 with negative magnetisation below the shadowed region, and probably the predicted phase transition happens exactly on the broken curve, i.e. on the mirror curve of the 'normal' phase transition curve with the formation of an infinite cluster. A preliminary analysis by Roussenq (1981) in $16\times16\times16$ Monte Carlo studies of the simple cubic lattice is compatible with some of our predictions.

To conclude, we mention that theorem 1 can be extended to many different situations, using mainly the proof of Aizenman *et al* (1980). In particular it holds for any equilibrium states constructed from a potential Φ with many-body forces and satisfying $\Sigma_{x\ni 0}\,\mathrm{diam}(x)|\Phi(x)|<\infty$, where $\mathrm{diam}(x)$ is the diameter of the set x of interacting particles. It holds also for a large class of systems with random interactions, and also for continuous models such as the Widom–Rowlinson models. Since such varied potentials lead to very different phase diagrams, one may expect from the above ideas a lot of different phase diagrams with percolative phase transitions with and without the formation of infinite clusters.

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